A CATEGORY ANALOGUE OF A THEOREM ON METRIC DENSITY

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ABSTRACT

The main result of this paper is to prove that for a subset B of the euclidean plane possessing the property of Baire, each point outside of some first category subset of B is a point of categorical directional density in all directions except, perhaps, a first category set of directions. This result is the category analogue of a result on metric directional densities derived by Bruckner and Rosenfeld and provides an affirmative answer to a question raised by Ceder in the setting of set theory.

1. Introduction

A standard measure-theoretic theorem asserts that any measurable subset M of (say) the euclidean plane R_2 has the property that all points of M outside some set of measure zero are points of (two dimensional) density of M as well as points of (one dimensional) density in any preassigned direction θ [4: p. 129 and p. 298]. This theorem has an analogue in the setting of category: any subset B of R_2 possessing the property of Baire has the property that all points of B outside some set of the first category are points of (two dimensional) categorical density of B as well as points of (one-dimensional) categorical density in any preassigned direction θ (we give the relevant definitions and background material for this statement in Section 2 below). In each theorem the second part states that for a fixed direction θ, the exceptional set is small in the sense of measure or category. But it is not true that there necessarily is one such exceptional set that works simultaneously for all directions. Nonetheless, it was shown in [1] that, in the case of measure, there is a small (zero measure) exceptional set such that each point (x, y) not in this set is a density point in all but a small (zero measure) set of directions. The purpose of this

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article is to prove that the corresponding statement is valid in the category setting. The theorem we prove also improves a category type theorem obtained in [2].

2. Preliminaries

We begin with a few definitions and notational conventions which we shall use in the sequel.

A point p in a topological space T is called *a point of the first category of a set B* if there is a neighborhood U of p such that $U \cap B$ is a first category subset of T. The point p is called *a categorical density point of B* if p is a point of the first category of B', the complement of B. When $B \subset R_2$, we shall also use such phrases as "p is a categorical density point of B in the direction θ" to mean p is a categorical density point of $B \cap L$ where L is the line through p whose angle of inclination is θ, $0 \leqq \theta < \pi$. If the rectangular coordinates of p are (x, y) we shall write $L(x, y, \theta)$ for this line. If n is a positive integer, we shall write $I_n(x, y, \theta)$ to designate the interval of $L(x, y, \theta)$ given by

$$I_n(x,y,\theta) = \left\{(x + r\cos\theta, y + r\sin\theta): -\frac{1}{n} < r < \frac{1}{n}\right\}.$$

Recall that a set B has the property of Baire provided it can be represented in the form $B = (K \sim M) \cup (M \sim K)$ where K is closed and M is of the first category. The collection of sets having the property of Baire forms a σ-algebra which contains all Borel sets and, in fact, all analytic sets. We shall use the following Proposition which follows immediately from the remarks on page 29 and theorem 15.4 of [3].

PROPOSITION. *If $B \subset R_2$ and has the property of Baire and $0 \leqq \theta < \pi$, then all points of B except for a set of the first category are categorical density points of B in the direction θ.*

To avoid unnecessarily lengthy phrases in the sequel, we shall often discuss the category of a set without explicitly mentioning whether this category is to be considered as a one, two, or three dimensional term. The context should make it clear which dimension is under consideration.

2. The categorical directional density theorem

THEOREM. *Let $B \subset R_2$ be a set having the property of Baire. There exists a first category subset A of B such that each point $(x, y) \in B \sim A$, is a categorical density point of $L(x, y, \theta)$ except for those θ in a first category subset of $[0, \pi)$.*

The proof of the theorem rests on two lemmas.

LEMMA 1. *The theorem is valid if B is closed.*

PROOF. Let $P = B \times [0,\pi)$. For each positive integer n, let

$Q_n = \{(x,y,\theta): B' \cap I_n(x,y,\theta)$ is of the first category in $L(x,y,\theta)\}$.

Since B is closed, B' is open so $B' \cap I_n(x, y, \theta)$ is empty whenever $(x, y, \theta) \in Q_n$. Let $Q = \bigcup_{n=1}^{\infty} Q_n$. Then Q consists exactly of those points (x,y,θ) for which (x,y) is a categorical density point of B in the direction θ. We show Q has the property of Baire by showing each Q_n is closed in P. Thus, let (x_j, y_j, θ_j) be a sequence of points in Q_n converging to a point (x_0, y_0, θ_0). Since $B' \cap I_n(x_j, y_j, \theta_j)$ is empty for each j, $B \cap I_n(x_j, y_j, \theta_j) = I_n(x_j, y_j, \theta_j)$ for each j. Since B is closed, it follows that $B \cap I_n(x_0, y_0, \theta_0) = I_n(x_0, y_0, \theta_0)$ so $B' \cap I_n(x_0, y_0, \theta_0)$ is empty and $(x_0, y_0, \theta_0) \in Q_n$. Thus Q_n is closed and hence Q has the property of Baire.

Now fix θ, $0 \leqq \theta < \pi$. By the proposition of Section 2, each point (x,y) not in some first category subset of B is a categorical density point of B in the direction θ. Thus, for each θ, the θ-section of $P \sim Q$ is a planar set of the first category. Since $P \sim Q$ has the property of Baire, it follows from a Fubini type category theorem (th. 15.4 of [3]) that $P \sim Q$ is a first category subset of P. Thus by the Kuratowski-Ulam theorem [3, th. 15.1], for all points (x,y) in B, outside some set of the first category, the (x,y) section of $P \sim Q$ is a first category subset of R_1. In other words, all points (x,y) outside some first category subset of B are categorical density points of B in all directions outside some first category set of directions. This proves Lemma 1.

LEMMA 2. *If B is a first category set in R_2, then each point of B is a first category point of B in all but a first category set of directions.*

PROOF. Suppose $(x,y) \in B$ but is not a first category point with respect to a set of directions Φ of second category. Assume, without loss of generality that $(x,y) = (0,0)$. Then, using polar coordinates to represent points of B other than $(0,0)$ we see for each $\theta \in \Phi$, $(r,\theta) \in B$ for a second category set of numbers r. It follows from [3, th. 15.1] that the set $\{(r,\theta): \theta \in \Phi,\ (r,\theta) \in B\}$ must be a second category set, considered as a subset of the (r,θ) plane. But this implies B is a second category subset of the (x,y) plane, contradicting our hypothesis.

We return to the proof of the theorem.

PROOF OF THEOREM. Since B has the property of Baire, we can write

$$B = (K \sim M) \cup (M \sim K)$$

where K is closed and M is a first category set. By Lemma 1, for all points (x, y) in K outside some first category subset of K, (x, y) is a categorical density point of K in all but a first category set of directions. By Lemma 2, each point is a first category point of M in all but a first category set of directions. It follows immediately that each point outside of some first category subset of B is a categorical density point of B in all but a first category set of directions.

REMARK 1. In [2], Ceder considered certain related questions with categorical density being replaced by a cardinality condition. In that setting it is, of course, impossible to obtain results completely analogous to our theorem, but theorems 4 and 7 of [2] are near analogues. In that connection, we mention that our theorem represents a generalization of Ceder's theorem 7. It also provides an affirmative answer to a question raised following theorem 7 of [2].

REMARK 2. We considered the case of R_2 for simplicity. Analogous results are valid in higher dimensions.

REMARK 3. Corresponding to density theorems about sets, there are differentiation theorems about functions. In the setting of measure theory, these theorems take the form of statements about approximate differentials or approximate partial (or directional) derivatives. Such theorems can be found in [4, p. 300] and [1]. It would be of interest to know whether corresponding results can be found in the category setting. One can define the categorical derivative of a real function of a real variable at a point x by lim $(f(y)-f(x))/(y-x)$ as $y \to x$, $y \in B$, if this limit exists for some set B having the property of Baire and having the point x as a categorical density point.

One can then consider such notions as categorical partial (or directional) derivatives or categorical total differentials and ask whether the existence on a residual set of one of these types of derivatives implies the existence on a residual set of the others. The analogous questions with regard to approximate derivatives have affirmative answers [4, p. 300], [1]. But the status of these questions does not seem to be known with respect to categorical derivatives.

References

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